QN.- State and Prove Leibnitz's theorem.

or, State and Prove Leibnitz's theorem for successive differentiation of product of function.

Ans.- Statement :- If u and v be two function of x which possesses nth derivative then

$(uv)_n = u_n v + {}^nc_1 u_{n-1} v_1 + {}^nc_2 u_{n-2} v_2 + \cdots + {}^nc_r u_{n-r} v_r + \cdots + {}^nc_n u v_n$

Proof :- Let $y = uv$

D.b.S.w.r.t. x, we have,

$y_1 = u_1 v + u v_1$

$y_2 = u_2 v + u_1 v_1 + u_1 v_1 + u v_2$

$= u_2 v + 2u_1 v_1 + u v_2$

$= u_2 v + {}^2c_1 u_1 v_1 + {}^2c_2 u v_2$

$y_3 = u_3 v + u_2 v_1 + 2(u_2 v_1 + u_1 v_2) + u_1 v_2 + u v_3$

$= u_3 v + 3u_2 v_1 + 3u_1 v_2 + u v_3$

$= u_3 v + {}^3c_1 u_2 v_1 + {}^3c_2 u_1 v_2 + {}^3c_3 u v_3$

In this way we see that the theorm is true for n = 1, 2, 3 - - -

Now, Let us assume that the theorem is true for n = m

$y_m = u_m v + {}^mc_1 u_{m-1} v_1 + {}^mc_2 u_{m-2} v_2 + \cdots + {}^mc_{r-1} u_{m-r+1} v_{r-1} + {}^mc_r u_{m-r} v_r + \cdots + {}^mc_m u v_m$

Again diff. both sides w.r.t. x, we have

$$y_{m+1} = u_{m+1}v + u_m v_1 + {}^mC_1(u_m v_1 + u_{m-1}v_2) + {}^mC_2(u_{m-r} + u_{m-2}v_3) + \cdots + {}^mC_{r-1}(u_{m-r+2}v_{r-1} + u_{m-r+1}\cdot v_r) + {}^mC_r(u_{m-r+1}v_r + u_{m-r}v_{r+1}) + \cdots + {}^mC_m(u_1 v_m + u v_{m+1})$$

$$y_{m+1} = u_{m+1}v + (1+{}^mC_1)u_m v_1 + ({}^mC_1 + {}^mC_2)u_{m-1}v_2 + \cdots + ({}^mC_{r-1} + {}^mC_r)u_{m+r+1}v_r + \cdots + {}^mC_m\, u v_{m+1}$$

$\because$ $1 = {}^mC_0$, ${}^mC_m = {}^{m+1}C_{m+1}$, ${}^mC_0 + {}^mC_1 = {}^{m+1}C_1$, ${}^mC_1 + {}^mC_2 = {}^{m+1}C_2$, ${}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r$

$$y_{m+1} = u_{m+1}v + {}^{m+1}C_1 u_m v_1 + {}^{m+1}C_2 u_{m-1}v_2 + \cdots + {}^{m+1}C_r u_{m-r+1}\cdot v_r + \cdots + {}^{m+1}C_{m+1} u v_{m+1}$$

In this way we see that the theorem is also true for $n = m+1$

Hence, the theorem is true for any integral +ve value of n.

— o —

Exercise:-1

① QN.→ If $y = A\sin mx + B\cos mx$, Prove that $y_2 = -m^2 y$.

Ans.→ $\because y = A\sin mx + B\cos mx$

D.b.s.w.r.t. x, we have

$y_1 = Am\cos mx + B\times(-m\sin mx)$

$y_1 = Am\cos mx - Bm\sin mx$

Again D.b.s.w.r.t. x, we have,

$y_2 = -Am^2\sin mx - Bm^2\cos mx$

$= -m^2(A\sin mx + B\cos mx)$

$y_2 = -m^2 y$ proved.

② QN.→ If $y = Ae^{mx} + Be^{-mx}$, Prove that $y_2 = m^2 y$.

Ans.→ $\because y = Ae^{mx} + Be^{-mx}$

D.b.s.w.r.t. x, we have

$y_1 = Am\,e^{mx} - Bme^{-mx}$

Again diff. b.s.w.r.t. x, we have

$y_2 = Am^2 e^{mx} + Bm^2 e^{-mx}$

$= m^2(Ae^{mx} + Be^{-mx})$

$y_2 = m^2 y$ proved

④ QN.→ If $p = a^2\cos^2\theta + b^2\sin^2\theta$, Show that

$$p + \frac{d^2p}{d\theta^2} = 2a^2 + 2b^2 = 3p.$$

Ans. $\rightarrow$ $\because p = a^2\cos^2\theta + b^2\sin^2\theta$

D. b. S. w. r. t. θ, we have

$$\frac{dp}{d\theta} = a^2 \times 2\cos\theta \times -\sin\theta + b^2 \, 2\sin\theta \cdot \cos\theta$$

$$= 2\sin\theta \cdot \cos\theta\,(b^2 - a^2)$$

$$\frac{dp}{d\theta} = (b^2 - a^2)\sin 2\theta$$

Again, diff. both sides, w. r. t. θ, we have

$$\frac{d^2p}{d\theta^2} = (b^2 - a^2) \times 2\cos 2\theta$$

$$= 2(b^2 - a^2)\cos 2\theta$$

$$L.H.S. = p + \frac{d^2p}{d\theta^2}$$

$$= a^2\cos^2\theta + b^2\sin^2\theta + 2(b^2 - a^2)(\cos^2\theta - \sin^2\theta)$$

$$= a^2\cos^2\theta + b^2\sin^2\theta + 2(b^2 - a^2)\cos^2\theta - 2(b^2 - a^2)\sin^2\theta$$

$$= a^2\cos^2\theta + b^2\sin^2\theta + 2b^2\cos^2\theta - 2a^2\cos^2\theta - 2b^2\sin^2\theta + 2a^2\sin^2\theta$$

$$= 2b^2\cos^2\theta - a^2\cos^2\theta - b^2\sin^2\theta + 2a^2\sin^2\theta$$

$$= 2(a^2\sin^2\theta + b^2\cos^2\theta) - (a^2\cos^2\theta + b^2\sin^2\theta)$$

$$= 2(a^2(1 - \cos^2\theta) + b^2(1 - \sin^2\theta) - p$$

$$= 2(a^2 - a^2\cos^2\theta + b^2 + b^2\sin^2\theta) - p$$

$$= 2(a^2 + b^2 - (a^2\cos^2\theta + b^2\sin^2\theta) - p$$

$$= 2(a^2 + b^2 - p) - p$$

$$2a^2 + 2b^2 - 2p - p$$

$= 2a^2 + 2b^2 - 3p$

$\therefore L.H.S. = R.H.S.$

(8) QN.→ If $y = a\sin(\log x)$, prove that $x^2y_2 + xy_1 + y = 0$

Ans.→ $\because y = a\sin(\log x)$

D.b.S. w.r.t. x, we have,

$y_1 = a\cdot\cos(\log x) \times \frac{1}{x}$

$xy_1 = a\cos\log x$

Again D.b.S. w.r.t. x, we have,

$x\cdot y_2 + y_1 \cdot 1 = a \times -\sin(\log x) \times \frac{1}{x}$

$xy_2 + y_1 = \frac{-a\sin(\log x)}{x}$

or, $xy_2 + y_1 = -\frac{y}{x}$

or, $x^2y_2 + xy_1 = -y$

or, $x^2y_2 + xy_1 + y = 0$ proved.

(9) QN.→ If $x = \cos(\log y)$, show that $(1-x^2)y_2 - xy_1 = y$

Ans.→ $x = \cos(\log y)$

$\log y = \cos^{-1}x$

D.b.S. w.r.t. x, we have

$\frac{1}{y} \times y_1 = -\frac{1}{\sqrt{1-x^2}}$

$y_1 \cdot \sqrt{1-x^2} = -y$

Squaring both sides, we have

$y_1^2(1-x^2) = y^2$

Again diff. b.s. w.r.t. x, we have

$2y_1 \cdot y_2(1-x^2) + y_1^2 x - 2x = 2yy_1$

$y_2(1-x^2) - xy_1 = y$ proved.

(10) QN.→ If $y = a\cos\log x$, prove that $x^2 y_2 + xy_1 + y = 0$.

Ans.→ $\because y = a\cos\log x$

D.b.s. w.r.t. x, we have

$y_1 = a \times -\sin(\log x) \times \frac{1}{x}$

$xy_1 = -a\sin\log x$

Again D.b.s. w.r.t. x, we have

$x \cdot y_2 + y_1 \cdot 1 = -a \times \cos(\log x) \times \frac{1}{x}$

$xy_2 + y_1 = \frac{-a\cos(\log x)}{x}$

or, $xy_2 + y_1 = \frac{-y}{x}$

or, $x^2 y_2 + xy_1 = -y$

or, $x^2 + y_2 + y = 0$ proved.